MAC-CPTM Situations Project

Situation 44: Zero Exponents

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Prompt

In an Algebra I class, a student questions the claim that $a^0 = 1$ *for all non-zero real number values of a*. The student asks, "How can that be possible? I know that a^0 is a times itself zero times, so a^0 must be zero."

Commentary

The succinct and mathematically correct answer to the student's question presented in the prompt is that a^0 is defined to be 1 for specific values of a. The arguments presented in the foci establish why this definition makes sense mathematically and why defining a^0 in such a way allows us to be consistent with other mathematical facts. The issue that the student raises in the prompt may be due to viewing a^0 as a numerical value. However, the broader perspective is that what matters is not the numerical value of the expression a^x but rather the characteristics properties (such as continuity) when one thinks of $y = a^x$ as a function.

Mathematical Foci

Mathematical Focus 1

The definition of $f(x) = a^x$ can be extended from having a domain of only non-zero integers to a domain of all real numbers.

The student appears to be drawing on a definition of exponents that is applicable only for exponents that are positive, whole numbers. When the values used as exponents are expanded the following are taken as part of the definition of exponent:

i) $a^0 = 1$ where a is any real number not equal to zero

ii) $a^{-n} = \dfrac{1}{a^n}$ when $n > 0$, and a is any real number not equal to zero

iii) $a^{m/n} = \sqrt[n]{a^m}$ where m is an integer, n is a positive integer, and a is a nonnegative real number.

The extension of the definition imposes restrictions on the values that may be used for the base, a.

Mathematical Focus 2

0^0 is defined to be an indeterminate form since the values of $\lim_{x\to 0} x^0$, $\lim_{x\to 0} 0^x$, and $\lim_{x\to 0} x^x$ are not consistent with each other.

The restriction on the definition $a^0 = 1$ that a cannot be equal to zero can be explained by examining the three functions: $f(x) = x^0$, $f(x) = 0^x$, and $f(x) = x^x$ as the value of x approaches zero. $\lim_{x\to 0^-}(x)^0 = 1$ and $\lim_{x\to 0^+}(x)^0 = 1$, thus providing some evidence that the value of 0^0 should be equal to one. However, $\lim_{x\to 0^-}(0)^x$ does not exist because the function does not exist for $x \le 0$ while $\lim_{x\to 0^+}(0)^x = 0$. Finally, although $\lim_{x\to 0^+}(x)^x = 1$, the $\lim_{x\to 0^-}(x)^x$ does not exist because the function is not continuous for $x < 0$. We also know that 0^n=0. If a=0, then $\dfrac{a^n}{a^n} = \dfrac{0}{0}$, is an indeterminate form.

Mathematical Focus 3

$a^0 = 1$ for $a \neq 0$ can be explained by using properties of exponents.

This scenario can be explained by using the multiplication and division properties of exponents. Using the division property of exponents, $\frac{a^n}{a^n}$ is equivalent to a^{n-n} or a^0, where n is any non-zero real number. We know that $\frac{a^n}{a^n} = 1$ because of the multiplicative identity field property. Therefore, because of the transitive property, a^0 must equal 1, that is, $1 = \frac{a^n}{a^n} = a^{n-n} = a^0$.

Alternatively, a fundamental property of exponentiation is that $a^{n+m} = a^n \cdot a^m$, for $a \neq 0$. Now consider the following,

$a^n = a^{n+0} = a^n \cdot a^0$ and thus, $a^0 = 1$.

Mathematical Focus 4

Defining $a^0 = 1$ *for* $a \neq 0$ *is consistent with the multiplicative relationship between successive terms in the sequence* $\{a^n\}$ *, where n is an integer and a is a non-negative real number.*

It could be helpful to look at a pattern involving the recursive nature of exponential growth in order to explore this question. First consider a specific example using exponents with base 4.

$$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$$
$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$
$$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$$
$$4^0 = ?$$
$$4^1 = 4 = 4$$
$$4^2 = 4 \cdot 4 = 16$$
$$4^3 = 4 \cdot 4 \cdot 4 = 64$$

As the exponent increases by 1, each successive term can be obtained by multiplying the preceding term by 4. That is, $4^{n+1} = 4 \cdot 4^n$. In order for this recursive pattern to hold for all integer values of *n*, then it seems that 4^0 should be equal to 1. It is important to note here that this pattern is developed using only integer values for the value represented by *n*. This pattern still holds if the exponents considered are non-integers. Consider the following:

$$4^{-\frac{12}{5}} = \frac{1}{4^{\frac{12}{5}}}$$
$$4^{-\frac{7}{5}} = \frac{1}{4^{\frac{7}{5}}} = \frac{1}{4^{\frac{12}{5}-1}} = \frac{1}{4^{\frac{12}{5}}} \cdot 4 = 4^{-\frac{12}{5}} \cdot 4$$
$$4^{-\frac{2}{5}} = \frac{1}{4^{\frac{2}{5}}} = \frac{1}{4^{\frac{7}{5}-1}} = \frac{1}{4^{\frac{7}{5}}} \cdot 4 = 4^{-\frac{7}{5}} \cdot 4$$
$$4^{\frac{3}{5}} = 4^{-\frac{2}{5}+1} = 4^{-\frac{2}{5}} \cdot 4^1$$
$$4^{\frac{8}{5}} = 4^{\frac{3}{5}+1} = 4^{\frac{3}{5}} \cdot 4^1$$
$$4^{\frac{13}{5}} = 4^{\frac{8}{5}+1} = 4^{\frac{8}{5}} \cdot 4^1$$

The primary reason to set up the pattern using only integer values for the exponents is to examine 4^0 as a part of a sequence of numbers written as 4^n, where *n* increases by 1.

This pattern can be generalized to all positive values of a. Consider the following table.

$$a^{-3} = \frac{1}{a^3}$$
$$a^{-2} = \frac{1}{a^2}$$
$$a^{-1} = \frac{1}{a^1}$$
$$a^0 = 1$$
$$a^1 = a$$
$$a^2 = a \cdot a$$
$$a^3 = a \cdot a \cdot a$$

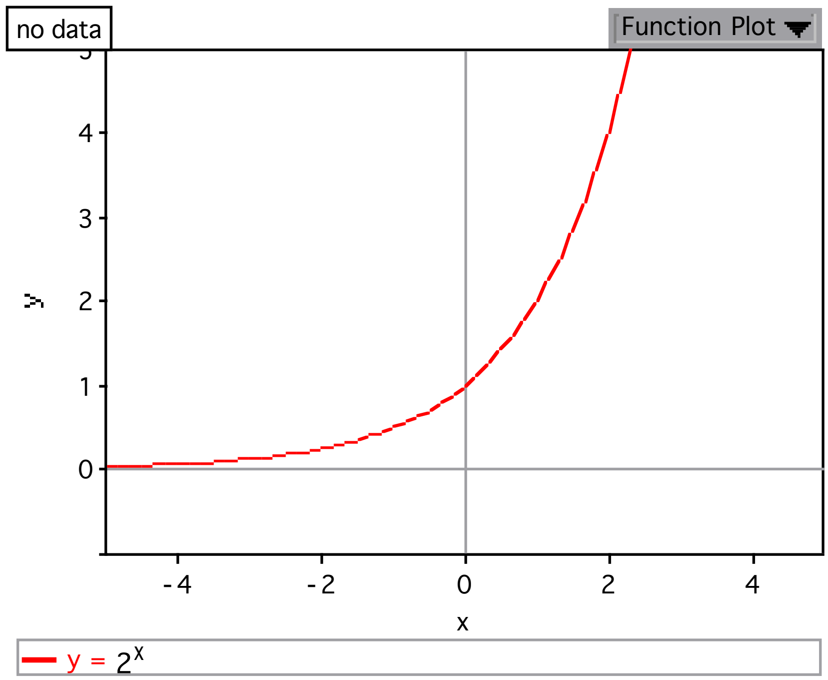
We can verify that the pattern holds by looking at a particular definition of a^n, where n is a whole number greater than or equal to 1. In this case, a^n is defined as the product of a multiplied to itself n times and a^{n+1} is a multiplied to itself $n+1$ times, which is the same as a times the product of a multiplied to itself n times. So, for all positive values of *a,* $a^{n+1} = a \cdot a^n$. When n = 0, then $a^{0+1} = a \cdot a^0$. Since $a^{0+1} = a^1 = a$, it follows then that a^0 must be equal to 1.

As in the specific case of $a = 4$ above, the general pattern of $a^{n+1} = a^n \cdot a$ does hold for any values for n. However, it is important to realize that this pattern will not hold for *all* values of a. For example $a^{\frac{1}{2}}$ is not a real number if a is any negative real number.

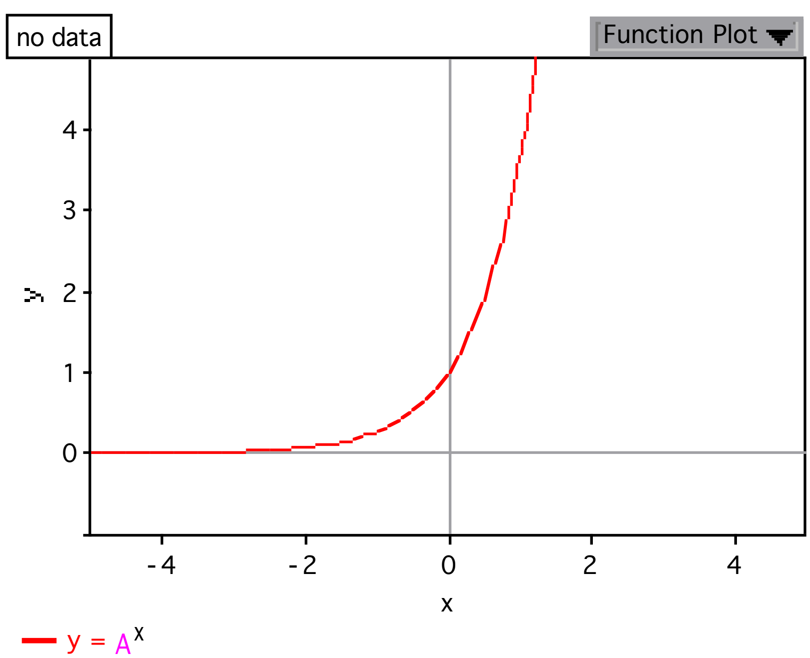
Mathematical Focus 5

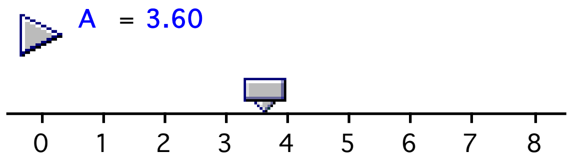
Defining $a^0 = 1$ *for* $a \neq 0$ *is consistent with pattern established by the graph of* $f(x) = a^x$ *for* $a > 0$ *and* $x \neq 0$.

Another approach to explore this problem is through a graphical representation of the function $y = a^x$ for various real values of a. The following graph depicts $y = 2^x$. The value of 2^0 appears to be equal to 1.

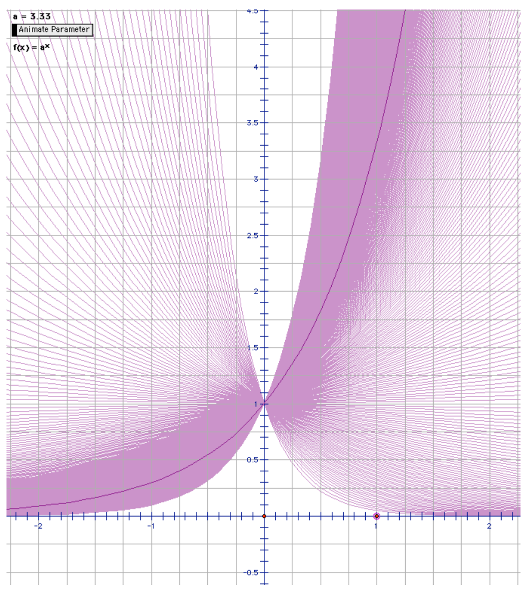


If we were to examine this more generally, the behavior of $y = a^x$ at $x = 0$ can be explored graphically for several positive values of a. This can be investigated dynamically using the slider feature in Fathom. The graph below represents $y = a^x$ where the value of a is indicated by the slider. In this case $a = 3.60$. As the value on the slider is changed, the graph is updated automatically to reflect the change. The value $a^0 = 1$ can be interpolated from the graph for any positive value of a



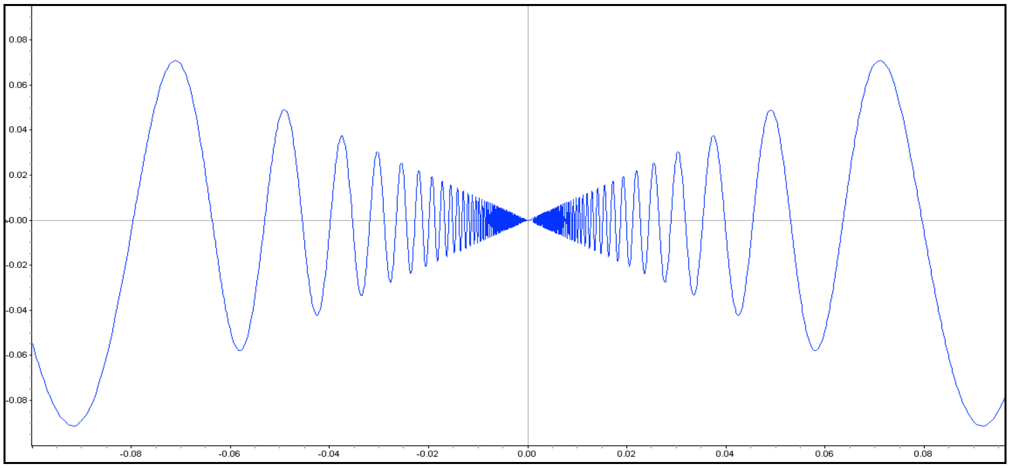


In fact, the point (0, 1) appears to be the common point for graphs of functions given by $y = a^x$, $a>0$, as we can see when the graphs of $y = a^x$ for positive values of a are traced in the following Geometer's Sketchpad sketch.



As mentioned in other foci, the assumptions placed on the value represented by a are important. In all the graphs given previously, it is assumed that a represents a positive non-zero real number. These graphs break down if a represents a non-zero negative number. For example, the graph of $y = (-2)^x$ is not a continuous and well-defined graph, since the function $y = (-2)^x$ is well-defined in the real number system only over a set of measure zero.

Other pertinent assumptions that underlie the above graphical argument are that the function $y = a^x$ is a well-defined and continuous function. The inferences drawn from the graphs above are based on these assumptions. If these assumptions are ignored, then the inferences drawn may be incorrect. For example, consider the graph of the function $y = x \cdot \sin\left(\frac{1}{x}\right)$ given below.



This graph looks smooth and the function seems to be well-defined. Therefore, the limit of the function as x approaches 0 may be estimated as 0. However, this inference is incorrect. This incorrect inference is easy to come to due to an issue of scale of the graph. $y = x \cdot \sin\left(\frac{1}{x}\right)$ is a function that oscillates with ever decreasing amplitude as x approaches 0. Although this amplitude approaches 0, it never attains the value of 0. Thus, regardless of how close to 0 the value of x is, the graph of the function will look like the given graph. Thus, the $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$ does not exist, as this value never approaches a single value.

Mathematical Focus 6

Defining $a^0 = 1$ *for* $a \neq 0$ *makes the function* $f(x) = a^x$ *continuous everywhere, for* $a > 0$.

Consider the function $f(x) = 2^x$. For this function to be continuous over all real x, $f(0)$ will have to be defined. To define $f(0)$, consider $\lim_{x \to 0}(2^x)$. To estimate $\lim_{x \to 0}(2^x)$ numerically, examine values of $f(x) = 2^x$ near $x = 0$.

x	2^x
-0.0004	0.99972278
-0.0003	0.999792077
-0.0002	0.99986138
-0.0001	0.999930688
0	?
0.0001	1.000069317
0.0002	1.000138639
0.0003	1.000207966
0.0004	1.000277297

As the values of x approach zero, the values of $f(x) = 2^x$ approach 1; therefore, it appears that $\lim_{x \to 0}(2^x) = 1$. This procedure can be expanded to all positive values for a.

To prove that $\lim_{x \to 0}(a^x) = 1$ (for positive values of a), show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|a^x - 1| < \varepsilon$ when $0 < |x - 0| < \delta$.

Let $\varepsilon > 0$ such that $|a^x - 1| < \varepsilon$. Now consider the following,

$$
\begin{aligned}
& |a^x - 1| < \varepsilon \\
\Rightarrow & -\varepsilon < a^x - 1 < \varepsilon \\
\Rightarrow & 1 - \varepsilon < a^x < 1 + \varepsilon
\end{aligned}
$$

<u>*Case 1*</u>: $0 < \varepsilon < 1$

Now,

$$
\begin{aligned}
& 1 - \varepsilon < a^x < 1 + \varepsilon \\
\Rightarrow & \ln(1 - \varepsilon) < x \cdot \ln a < \ln(1 + \varepsilon) \\
\Rightarrow & \frac{\ln(1 - \varepsilon)}{\ln a} < x < \frac{\ln(1 + \varepsilon)}{\ln a}
\end{aligned}
$$

In this case, choose $\delta = \max\left\{\left|\frac{\ln(1 - \varepsilon)}{\ln a}\right|, \left|\frac{\ln(1 + \varepsilon)}{\ln a}\right|\right\}$.

Therefore, $|x| < \delta$.

<u>*Case 2*</u>: $\varepsilon \geq 1$

In this case, choose $\delta = \left|\frac{\ln(1 + \varepsilon)}{\ln a}\right|$.

Therefore, $|x| < \delta$.

Thus, $\lim_{x \to 0}(a^x) = 1$.

Mathematical Focus 7

Defining $a^0 = 1$ for $a \neq 0$ allows us to extend the domain of $f(x) = x^0$ to non-zero complex numbers.

Thus far we have considered the value of a^0 only when a is a nonzero real number. We can also consider the value of this expression when a is a complex number not equal to zero. First, consider the case when $a = i$. Begin with the imaginary unit raised to an integer power. Using the definition of i, $i = \sqrt{-1}$, then the following hold:

$$
\begin{aligned}
i^{-4} &= 1 \\
i^{-3} &= i \\
i^{-2} &= -1 \\
i^{-1} &= -i \\
i^{0} &= ? \\
i^{1} &= i \\
i^{2} &= -1 \\
i^{3} &= -i \\
i^{4} &= 1 \\
i^{5} &= i
\end{aligned}
$$

The powers of the imaginary unit rotate around the unit circle on the complex plane. Therefore, since $i^1 = i$ and $i^4 = 1$, it follows that $i^0 = 1$.

We can also consider the value of any complex, non-zero number raised to the zero power by rewriting $(a+bi)^0$ in polar form and applying DeMoivre's theorem, $(r\cos\theta + ri\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$. Thus,

$$(r\cos\theta + ri\sin\theta)^0 = r^0(\cos(0*\theta) + i\sin(o*\theta)) = r^0(1+0).$$

Since r is a real, nonzero number, $r^0 = 1$ and we can conclude that $(a+bi)^0$ equals one providing a and b are not both equal to zero. Thus $a^0 = 1$ not only for real, non-zero values of *a* but for complex, non-zero values as well.

References

Lakoff, George; Nunez, Rafael; Where Mathematics Comes From: How the Embodied Mind Brings Math into Being; 2001

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